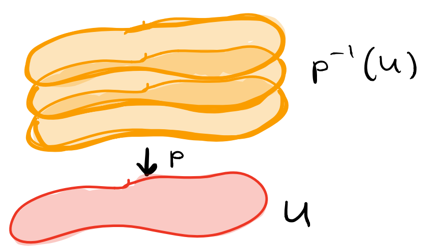
Covering Spaces

Our goal in this section is to prove our first big result about fundamental groups:

The fundamental group of S^1 is isomorphic to $\mathbb{Z}$.

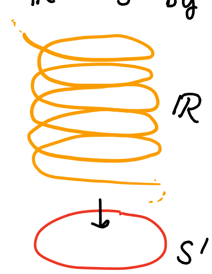
Def: Let $p: E \to B$ be continuous and surjective. p <u>evenly covers</u> an open set $U \subseteq B$ if we can write $p^{-1}(U) = \bigsqcup V_\alpha$, where V_α are disjoint open sets s.t. for each α, $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. The V_α are called <u>slices</u>.



If every point $b \in B$ has a neighborhood U that is evenly covered by p, then p is a <u>covering map</u>, and E is a <u>covering space</u> of B.
(B is called the <u>base</u> of the covering.)

Ex: Define the map $p: \mathbb{R} \to S^1$ by $p(x) = (\cos x, \sin x)$.

This is a covering map! Consider the point $(1,0) \in S^1$.



The open set

$U = \{(x,y) \mid x > 0\} \cap S^1$

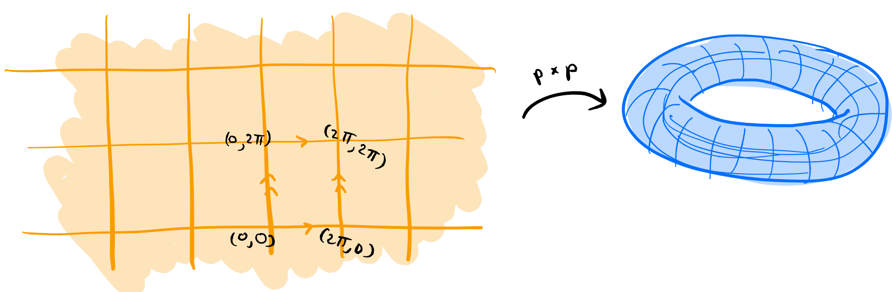
has preimage $p^{-1}(u) = \bigsqcup_{n \in \mathbb{Z}} \left(2\pi n - \frac{\pi}{2}, 2\pi n + \frac{\pi}{2}\right)$

Thm: If $p: E \to B$ and $q: E' \to B'$ are covering maps, then so is

$$p \times q : E \times E' \to B \times B'.$$

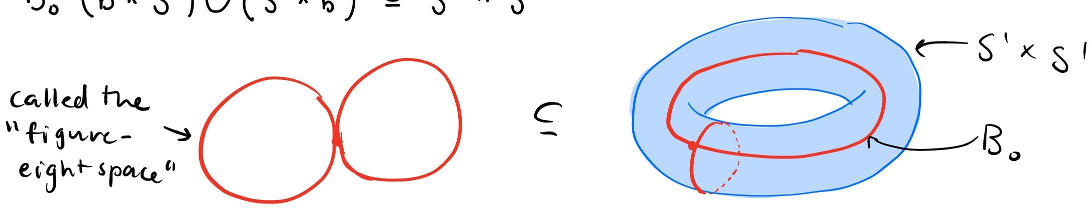
Pf: If $(b, b') \in B \times B'$, w/ corresponding evenly covered neighborhoods $U \ni b$ and $U' \ni b'$, then $p^{-1}(U \times U') = p^{-1}(U) \times p^{-1}(U')$ which is the union of open slices of the form $V_\alpha \times V'_\beta$, homeomorphic to $U \times U'$, as desired. $\square$

Example: Consider the torus $S^1 \times S^1$. Since $\mathbb{R}$ covers S^1, $\mathbb{R}^2$ covers $S^1 \times S^1$.



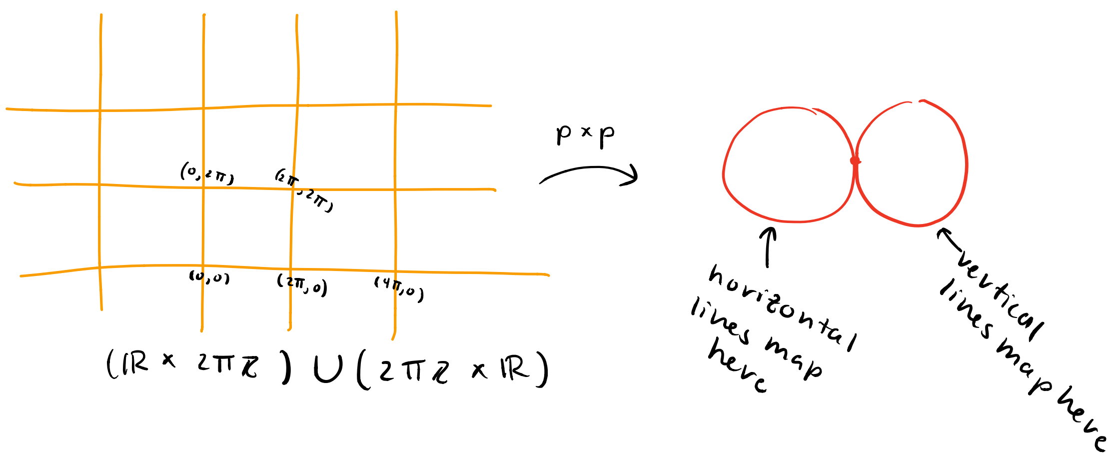
If we have a covering $p: E \to B$ and $B_0 \subseteq B$ a subspace, we also get a covering $p^{-1}(B_0) \to B_0$:

Ex: Let $b \in S^1$ be a point on the circle, and consider $B_0 = (b \times S^1) \cup (S^1 \times b) \subseteq S^1 \times S^1$



B_0 is the union of two circles w/ a point in common.

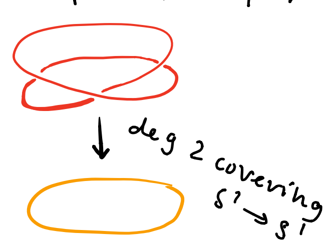
We obtain a covering space of B_0 by taking $(p \times p)^{-1}(B_0)$:



Ex: For any topological space X, we can take $f: \bigsqcup_\alpha X_\alpha \longrightarrow X$ where X_α is homeomorphic to X, and $f|_{X_\alpha}$ is a homeomorphism. f is a covering map.

Exercise: (on HW) If B is connected, and $p: E \to B$ is a covering map, then for any two points $x, y \in B$, the cardinality of the fibers $p^{-1}(x)$ and $p^{-1}(y)$ are the same. If the cardinality is $d < \infty$, d is called the degree of the covering.

Ex: Consider $S^1 \subseteq \mathbb{C}$ consisting of complex #s w/ $|z| = 1$. Then $p: S^1 \longrightarrow S^1$ given by $p(z) = z^n$ is an n-fold covering.



Lifting

Def: Let $p: E \longrightarrow B$ be a continuous function. If $f: X \longrightarrow B$ is another continuous function to B, then a lifting of f is a map $\tilde{f}: X \longrightarrow E$ s.t. $p \circ \tilde{f} = f$.

i.e.

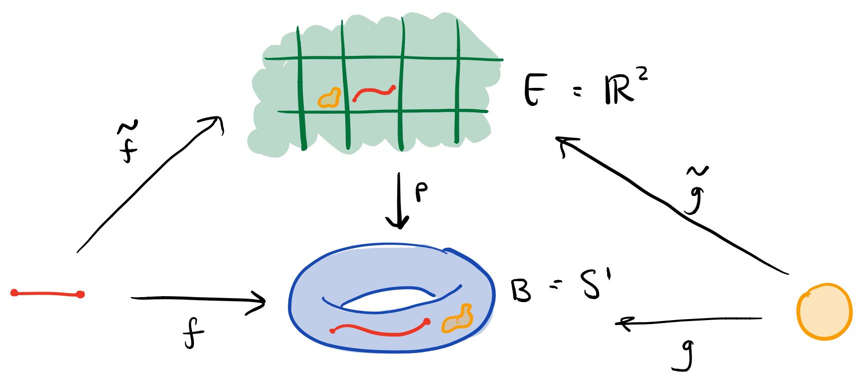
$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

If $p: E \to B$ is a covering map, then we can "locally" lift functions.

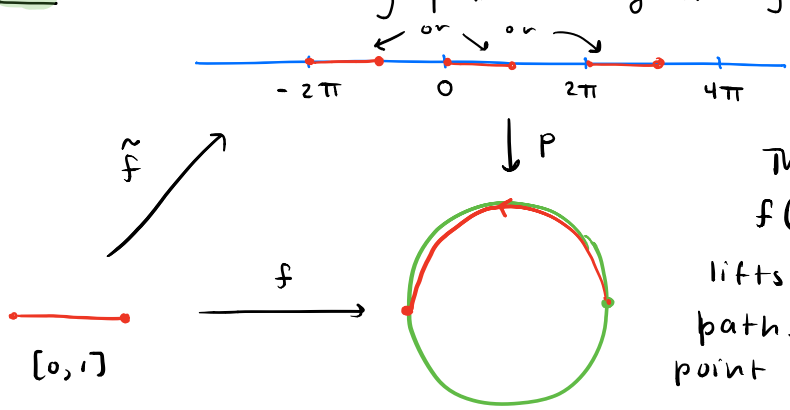
i.e. if $f(X) \subseteq U \subseteq B$ and U is evenly covered, then we can lift f to one of the sheets.

Ex:



We'll see that if $p: E \to B$ is a covering, paths and path homotopies on B can be lifted.

Ex: Consider the covering $p: \mathbb{R} \to S^1$ given by $p(x) = (\cos x, \sin x)$.



The path $f(s) = (\cos \pi s, \sin \pi s)$ lifts to many possible paths, depending which point 0 gets sent to.

Theorem: Let $p: E \to B$ be a covering map and $f: [0,1] \to B$ a path. Suppose $f(0) = b = p(e)$. Then there is a <u>unique</u> lifting $\tilde{f}: [0,1] \to E$

s.t. $f(0) = e$.

Pf: Cover B by open sets U which are each evenly covered by p. The preimages $f^{-1}(U)$ cover $[0,1]$.

Since $[0,1]$ is a compact metric space, the Lebesgue # Lemma tells us that $\exists$ some δ s.t. $(x, x+\delta) \subseteq f^{-1}(U)$, some U $\forall x$.

Thus, we can choose a subdivision $0 = s_0 < s_1 < \dots < s_n = 1$ s.t. $f([s_i, s_{i+1}])$ lies in one of the open sets.

Define $\tilde{f}(0) = e$. Assume we can define $\tilde{f}(s)$ for $0 \le s \le s_i$.

Define $\tilde{f}$ on $[s_i, s_{i+1}]$ as follows:

$f([s_i, s_{i+1}]) \subseteq U$, some open U evenly covered by p.

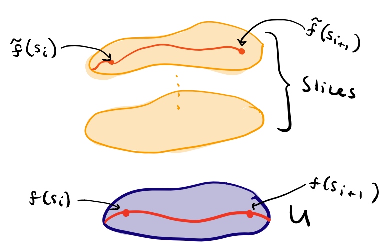
Let $V \subseteq p^{-1}(U)$ be the slice containing $\tilde{f}(s_i)$.

Then for $s \in [s_i, s_{i+1}]$ define $\tilde{f}(s) = p|_V^{-1}(f(s))$.

$p|_V$ is a homeomorphism, so $\tilde{f}$ is continuous on $[s_i, s_{i+1}]$. By induction,

we can define a continuous function $\tilde{f} : [0,1] \to E$.

$\tilde{f}$ is unique since for each s_i there was a unique slice containing $\tilde{f}(s_i)$. $\square$

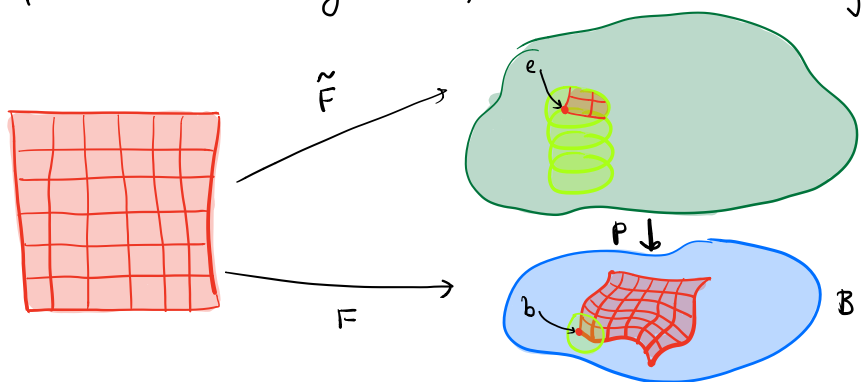


A similar result holds for maps from $I \times I$ $(I = [0,1])$:

Thm: Let $F: I \times I \to B$ be continuous w/ $F(0,0) = b_0$. Let $p: E \to B$ be a covering map w/ $p(e) = b$. There is a unique lifting of F

$$\tilde{F}: I \times I \to E \quad \text{s.t.} \quad \tilde{F}(0,0) = e.$$

Pf: The proof is just like the previous except we subdivide $I \times I$ into squares w/ side length $< \delta$, where δ is a lebesgue # for



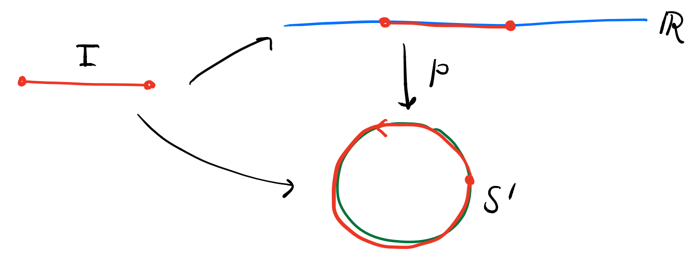
the preimage of the covering of $F(I \times I)$ by open sets that are evenly covered.

We then build the lifting square by square. $\tilde{F}$ is unique, because at each step there is a unique choice of slice that will make $\tilde{F}$ continuous. $\square$

Note that if F is a path homotopy from f to g, $\tilde{F}$ will be a path homotopy from $\tilde{f}$ to $\tilde{g}$: If $F(0,t) = b_0$, then $p(\tilde{F}(0,t)) = b_0$, so $\tilde{F}(0,t) \subseteq p^{-1}(b_0)$ which is a set of points w/ discrete topology. Thus $\tilde{F}(0,t)$ is a single point, and, similarly, so is $\tilde{F}(1,t)$.

What about lifting loops? Loops don't always lift to loops!

Ex:



However, since path lifting is unique, given a starting point, the endpoint is also unique. i.e., if $b_0 \in B$ and $e_0 \in E$ s.t. $p(e_0) = b_0$, then there is a function on sets

$$\varphi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

defined $\varphi([f]) = \tilde{f}(1)$, where $\tilde{f}$ is the lift of f s.t. $\tilde{f}(0) = e_0$

This is called the lifting correspondence.

Why is φ well-defined? If F is a path homotopy from f to g, $\tilde{F}$ is a path homotopy from $\tilde{f}$ to $\tilde{g}$, so $\tilde{f}(1) = \tilde{g}(1)$.

Ex: If we take the covering $\pi_1 : \mathbb{R} \to S_1$, $\varphi([f]) = 2\pi k$, where k depends on how many times f loops around S^1, and in which direction (CW or CCW).

Claim: If E is path connected, φ is surjective.

Pf: Let $e_1 \in p^{-1}(b_0)$, $g : [0,1] \to E$ a path from e_0 to e_1. Then $p \circ g$ is a loop at b_0 and $g = \widetilde{(p \circ g)}$. $\square$

When is φ bijective? When two paths starting and ending at e_0 and e_1, resp., always have homotopic images in B.

Theorem: If X is simply connected, any two paths f, g from x_0 to x_1 are path homotopic.

Pf: $f * \bar{g}$ is a loop at x_0, so $[f * \bar{g}] = [e_x]$

Thus, $[f] * [\bar{g}] * [g] = [e_x] * [g] \Rightarrow [f] = [g] \Rightarrow f \simeq_p g$. $\square$

Cor: If $p: E \to B$ is a covering and E is simply connected, $\varphi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is a bijection.

Pf: If $\varphi([f]) = \varphi([g])$, then $\tilde{f}$ and $\tilde{g}$ are paths starting at e_0 and ending at some e_1. Thus, $\tilde{f} \simeq_p \tilde{g}$, so $f = p \circ \tilde{f} \simeq_p p \circ \tilde{g} = g \Rightarrow [f] = [g]$. $\square$

Thm: $\pi_1(S^1) \cong \mathbb{Z}$.

Pf: Let $p: (\mathbb{R}, 0) \to (S^1, (1,0))$ be the covering $p(x) = (\cos(2\pi x), \sin(2\pi x))$. Then, since $\mathbb{R}$ is simply connected,

$$\varphi : \pi_1(S^1, (1,0)) \to p^{-1}((1,0)) = \mathbb{Z} \text{ is a bijection.}$$

We just need to show φ is a homomorphism.

Let $[f], [g] \in \pi_1(S^1)$ and $\varphi([f]) = n$, $\varphi([g]) = m$.

Define a new path $h: [0,1] \to \mathbb{R}$ by $h(s) = n + \tilde{g}(s)$,
where $\tilde{g}$ is the lifting of g s.t. $\tilde{g}(0) = 0$.

Then $\tilde{f} * h$ is defined and is a path from 0 to $m+n$.

But $p(h(s)) = p(n + \tilde{g}(s)) = p(\tilde{g}(s)) = g(s)$, since p is periodic w/ period 1.

Thus, $\tilde{f} * h$ is the lifting of $f * g$ beginning at 0.

$\Rightarrow \varphi([f]*[g]) = n+m$, as desired. $\square$

Exercise: Using a similar argument, we can show $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$.